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# On the evolution equations for ideal magnetohydrodynamics in curved spacetime

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**Abstract** We examine the problem of the construction of a first order symmetric hyperbolic evolution system for the Einstein-Maxwell-Euler system. Our analysis is based on a $1+3$ tetrad formalism which makes use of the components of the Weyl tensor as one of the unknowns. In order to ensure the symmetric hyperbolicity of the evolution equations implied by the Bianchi identity, we introduce a tensor of rank 3 corresponding to the covariant derivative of the Faraday tensor. Our analysis includes the case of a perfect fluid with infinite conductivity (ideal magnetohydrodynamics) as a particular subcase.

**Keywords** Magnetohydrodynamics, initial value problem

## 1 Introduction

The present article is concerned with the construction of a first order system of quasilinear hyperbolic evolution equations describing a charged ideal perfect fluid coupled to the Einstein field equations —the *Einstein-Maxwell-Euler system*. As the fluid is charged, one needs to bring into consideration the Maxwell equations to describe the electromagnetic field produced by the charged fluid flow. This system contains as an important subcase that of the so-called *ideal magnetohydrodynamics (MHD)* —which applies applies to the study of plasma in situations where a fluid subjected to significant magnetic fields.

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As it is well known, General Relativity admits initial value problem formulation whereby one prescribes certain initial data on a 3-dimensional hypersurface, and one purports to reconstruct the spacetime associated to this initial data —a so-called *Cauchy problem*. The formulation of an initial value problem is a necessary starting point for the construction of numerical solutions to, say, the Einstein-Maxwell-Euler system. The importance of a initial value formulation of the system under consideration is not restricted to numerical considerations, it is also a natural starting point for a wide variety of analytical studies of the qualitative properties of the solutions to the equations. *In this article we approach the construction of the evolution equations of the Einstein-Maxwell-Euler system from the point of view of mathematical Relativity*. Hence, the amenability of our analysis to analytic considerations takes precedence over numerical considerations of implementability. Examples of qualitative aspects of the solutions to, say, the Einstein-Maxwell-Euler system requiring an suitable initial value formulation are the discussion of local and global existence problems and the analysis of the stability of certain reference solutions. Our initial value formulation of MHD will be used elsewhere to discuss the stability of certain spherically symmetric configurations using the methods of [1].

In order to formulate an initial value problem for the Einstein-Maxwell-Euler system, one has to break the covariance of the equations by introducing coordinates and choosing a preferred timelike direction in the spacetime. This choice of coordinates and timelike direction is usually known as a *gauge choice*. In matter models which contain fluids, there is a natural timelike direction —that given by the fluid flowlines. The reasons behind the need of a gauge choice to formulate an initial value problem are technical: the machinery of the theory of partial differential equations does not apply to tensorial objects. As a consequence, the discussion of initial value problems in General Relativity is inherently gauge dependent.

The motivation behind the gauge choice procedure discussed in the previous paragraph is to deduce, from our basic set of (tensorial) equations, a closed[1] subsystem of evolution equations which is in what is called *symmetric hyperbolic* form —see e.g. [2] for precise definitions. This procedure is sometimes called *hyperbolic reduction*. Once a symmetric hyperbolic system of equations has been obtained then the machinery of the theory of hyperbolic equations applies and ensures that for suitable data prescribed on an initial hypersurface, the evolution equations have a local solution (i.e. a solution that exists at least for for a small interval of time) and that the solutions thus obtained is unique. The same theory also ensures that the solutions obtained from the Cauchy problem depend continuously on the values of initial data. An initial value problem with the properties described in the previous paragraph is said to be *well-posed*. It is important to point out that if a certain reduction procedure does not lead an hyperbolic system, this does not imply

[1] *Closed* in this context means that there should be as many equations as unknowns.

that the resulting problem is not well posed —this this would require a much more detailed analysis. Symmetric hyperbolicity is just a necessary condition for well-posedness.

The purpose of the present article is to show that the Einstein-Maxwell-Euler system admits a reduction procedure leading to a system of symmetric hyperbolic evolution equations. This system of equations is of first order. As a consequence of our construction one automatically obtains a local existence and uniqueness result for the system under consideration.

The hyperbolic reduction procedure described in the present article borrows from the discussion of the evolution equations for the Einstein-Euler system by H. Friedrich in [3] —see also [2,4] and [5]. In this reference a Lagrangian gauge was used to undertake the hyperbolic reduction and to obtain the desired evolution equations. The term *Lagrangian* means, in this context, that one uses the flow lines of the fluid to construct the preferred time direction on which the hyperbolic reduction is based. As it will be seen in the main body of the article, this choice has important technical advantages. The central equation in this discussion is the *Bianchi identity*. It provides evolution equations for the components of the Weyl tensor. The addition of electromagnetic interactions to Friedrich's system destroys, in principle, the symmetric hyperbolic nature of the evolution equation as derivatives of the Faraday tensor enter into the principal part of the Bianchi evolution equations. This difficulty can be handled by the introduction of a new field unknown corresponding to the derivative of the Faraday tensor for which suitable field and evolution equations can be obtained —this idea is borrowed from the analysis of the conformal Einstein-Maxwell system of [6]. The system of equations considered are similar in spirit to those frame formalism equations of [7].

The evolution equations for the Einstein-Maxwell-Euler system were first considered from the perspective from the Cauchy problem in [8] —see also [9]. This analysis makes use of a *traditional* formulation of the Einstein field equations in which the gravitational field is described by the metric tensor, and uses wave coordinates to obtain non-linear wave equations for the metric tensor. In this approach, the resulting evolution system is of mixed order. As a consequence, the so-called *Leray theory* needs to be used —see [9]. As pointed out in that reference, a first order formulation of the problem does not necessarily improve the analytic results that can be obtained with a mixed order one. However, first order formulations may be preferred in numerical computations and in the analysis of the motion of isolated bodies —notice, however, the local existence result for isolated bodies consisting of dust (charged and uncharged) given in [10] which is based on a mixed order formulation. In any case, to the best of the author's knowledge, there is no frame formulation of the Einstein field equation with is used in current numerical simulations of the Einstein field equations. From our particular point of view, the value of first order formulations of the Einstein-Euler system like the one being used here lies in their amenability to a detailed analysis of their linear and non-linear stability —see [11,12]. Alternative discussions of the problem of the

well-posedness of the evolution equations of the Einstein-Maxwell-Euler system and of magnetohydrodynamics can be found in, for example, [13,14,15, 16,17], and [18].

There are several formulations of the evolution equations of the Einstein-Maxwell-Euler system geared towards its use in numerical computations. Without the aim of being exhaustive, we notice in particular [19,20,21] —a good point of entry to the extensive literature in this respect is the review [22]. A peculiarity of the systems given in [19,20,21] is their *explicit use of the conditions of ideal magnetohydrodynamics* to construct their system. Hence, they may require some modifications if one is to use them to model more general types of electromagnetic fields. The question of the hyperbolicity of these systems is by no means direct and has to be analysed with certain amount of care. In particular the reference [19] makes use of an ADM formulation for the *Einstein part* of the system which is known to have problems concerning hyperbolicity —see e.g. [4]. This problem has been addressed in [20,21] by using a so-called BSSN formulation for the geometric part —see e.g. [23] for further details. The BSSN equations are first order in time and second order in the spatial coordinates and are known to satisfy, in the vacuum case, a certain notion of hyperbolicity (*strong hyperbolicity*) under certain additional conditions —see e.g. [24]. Local existence and uniqueness of solutions to strong hyperbolic systems follows under some further assumptions —see e.g. [25]. However, because of the mixed order nature of the system, it is not directly obvious that this hyperbolicity property is preserved if one extends the vacuum Einstein system to the Einstein-Maxwell-Euler one despite the *lower order coupling* of gravity to the charged fluid. To the best of our knowledge this issue has not been addressed in the literature. A discussion on the caveats of this type of reasoning can be found, for example, in Section 8.5 of [25].

It should be pointed out that the evolution equations discussed in the present article are only suited to the discussion of unbounded self-gravitating charged fluids. In order to accommodate an initial boundary value problem, further structure needs to be verified in the equations. For example, it is not known whether our evolution equations are compatible with the class of maximally dissipative boundary conditions used in, say [26], to show the well-posedness of the initial boundary problem for the vacuum Einstein field equations. In any case, it is interesting to point out that the analysis of [26] makes use of a frame formalism similar to the one used in the present article.

Symmetric hyperbolicity, despite being a technical mathematical notion has a deep physical significance. Moreover, the question of the well-posedness of the evolution equations of the Einstein-Maxwell-Euler system is a problem that touches upon many aspects of current theoretical and numerical analysis of physical phenomena. Particular examples of these systems are given by by accretion disks around compact objects. Due to the complexity of the underlying equations, the analysis of the stability and the shape of disks and flows is often addressed by numerical methods. Further applications of charged perfect fluids in General Relativity can be found in, for example, [22,17,27,28].

Numerical simulations of magnetohydrodynamical systems with astrophysical applications in view have been a topic of interest in recent years —see for example [29,30,31,32,33].

The present article is structured as follows: The tetrad formalism used in this article is briefly reviewed in Sec. 2. In Sec. 3 we write and discuss the relativistic equations describing a charged perfect fluid. General remarks concerning the reduction procedure to obtain suitable evolution equations are given in Sec. 4; the resulting evolution equations are discussed in Sec. 4.2 and the subsequent sections. The auxiliary field describing the covariant derivative of the Faraday tensor is presented in Sec. 4.3. A summary of the evolution equations is given in Sec. 4.6. The case of an infinitely conductive fluid (MHD) is briefly explored in Sec. 4.7. Some concluding remarks are given in Sec. 5. Finally, in Appendix A we briefly discuss certain issues concerning homentropic flows and their relation to the equation of state of the perfect fluid.

## 2 Tetrad formalism

As mentioned in the introduction, our discussion of the Einstein-Maxwell-Euler system will be based on a frame formalism. The purpose of the present section is, mainly, to fix the conventions to be used in the rest of the article.

Let $(\mathcal{M}, g_{\alpha\beta})$ denote a spacetime. Our discussion will consider the two possible conventions for the signature of the metric tensor $g_{\alpha\beta}$. We introduce *frame fields* $\{e_a\}_{a=0,\ldots,3}$ on the spacetime $\mathcal{M}$ satisfying $g_{ab} \equiv g(e_a, e_b) = \eta_{ab} = \epsilon\ \mathrm{diag}(1,-1,-1,-1)$ and $\epsilon = \pm 1$ —see [34,7]. We denote by $(\omega^a)$ the corresponding *dual basis* (cobasis)[2]. The frame fields $e_a$ and the cobasis $\omega^a$ are expressed in terms of a local coordinate basis as

$$e_a = e_a{}^{\alpha}\partial_\alpha, \qquad \omega^a = \omega^a{}_\alpha \mathrm{d}x^\alpha. \tag{1}$$

Thus,

$$\omega^a{}_\alpha e_b{}^\alpha = \delta^a{}_b, \qquad \omega^a{}_\alpha e_a{}^\beta = \delta_\alpha{}^\beta. \tag{2}$$

The metric tensor and its inverse can be written as can be written in terms of $\omega^a$ and $e_a$ as

$$g_{\alpha\beta} = \eta_{ab}\omega^a{}_\alpha\omega^b{}_\beta, \qquad g^{\alpha\beta} = \eta^{ab}e_a{}^\alpha e_b{}^\beta. \tag{3}$$

The *coefficients of the commutator of the elements of the tetrad*, $D_{ab}{}^c$, are defined by

$$[e_a, e_b] \equiv D_{ab}{}^c e_c = \left(e_a{}^\alpha\partial_\alpha e_b{}^\beta - e_b{}^\alpha\partial_\alpha e_a{}^\beta\right)\partial_\beta. \tag{4}$$

The *connection components* (Ricci rotation coefficients), $\Gamma_b{}^a{}_c$, of the tetrad $e_a$ are defined by the relations

$$\nabla_a e_b = \Gamma_a{}^c{}_b e_c, \qquad \nabla_a\omega^b = -\Gamma_a{}^b{}_c\omega^c. \tag{5}$$

[2] Latin letters $a$ are the tetrad indices while Greek letters $\alpha$ denotes the tensorial character of each tensor (spacetime indices). Latin letters $(i,j,k,q,p)$, run in $\{1,2,3\}$, denoting quantities in the tetrad space.

In particular, since $e_a(\eta_{bc}) = 0$, one has the symmetry $\Gamma_{a(bc)} = 0$. The *Riemann tensor*, $R^a{}_{bcd}$, satisfying the *Ricci identity*

$$R^a{}_{bcd}\omega_a = \nabla_c \nabla_d \omega_a - \nabla_d \nabla_c \omega_a - \nabla_{[c,d]}\omega_b, \tag{6}$$

is given in terms of the connection coefficients $\Gamma_b{}^a{}_c$ by

$$R^a{}_{bcd} = e_c\left(\Gamma_d{}^a{}_b\right) - e_d\left(\Gamma_c{}^a{}_b\right) - \Gamma_e{}^a{}_b(\Gamma_c{}^e{}_d - \Gamma_d{}^e{}_c) + \Gamma_c{}^a{}_e\Gamma_d{}^e{}_b - \Gamma_d{}^a{}_e\Gamma_c{}^e{}_b. \tag{7}$$

In the sequel we will make use of the following decomposition of the Riemann tensor

$$R_{abcd} = C_{abcd} + \left\{g_{a[c}S_{d]b} - g_{b[c}S_{d]a}\right\}, \tag{8}$$

in terms of the *Schouten tensor* $S_{ab}$

$$S_{ab} \equiv R_{ab} - \frac{1}{6}Rg_{ab}, \tag{9}$$

and the *Weyl tensor*

$$C_{abcd} \equiv R_{abcd} + (g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{1}{3}\ R\ g_{a[c}g_{d]b}. \tag{10}$$

In the previous expressions $R_{ab} \equiv R^c{}_{acb}$ denotes the *Ricci tensor* of $g_{\alpha\beta}$, and $R \equiv g^{ab}R_{ab}$ its *Ricci scalar*. The components of the curvature tensor satisfy the *Bianchi identity*

$$R^a{}_{b[cd;e]} = 0. \tag{11}$$

Now, from the contracted Bianchi identities for the *Einstein tensor*, $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$, satisfying

$$\nabla^a G_{ab} = 0, \tag{12}$$

we infer that

$$F_{abc} \equiv \nabla_d F^d{}_{abc} = 0, \tag{13}$$

where $F_{abcd}$ denotes the so-called *Friedrich tensor*

$$F_{abcd} \equiv C_{abcd} - g_{a[c}S_{d]b}. \tag{14}$$

Taking the Hodge dual of Eq. (13) with respect to the index pair $(c,d)$, we obtain an equation similar to (13) for the tensor $\tilde{F}_{abcd}$ defined by

$$\tilde{F}_{abcd} \equiv C^*_{abcd} + \frac{1}{2}S_{pb}\epsilon^p{}_{acd}, \tag{15}$$

where $\epsilon_{abcd}$ denotes the components of the *completely antisymmetric Levi Civita tensor* with respect to the frame $e_a$ and $C^*_{abcd} \equiv \frac{1}{2}C_{abef}\epsilon^{ef}{}_{cd}$.

## 3 Ideal magnetohydrodynamics (MHD) in curved spacetime

In this section we introduce the basic equations describing a relativistic charged perfect fluid coupled gravity —the Einstein-Maxwell-Euler system. The notation and conventions used are based on those of [3] and [27,28].

We start by considering the following *Einstein equations*

$$G_{ab} = \kappa T_{ab}, \tag{16}$$

with an *energy-momentum tensor*, $T_{ab}$, which can be split as $T_{ab} \equiv T^{\mathrm{f}}_{ab} + T^{\mathrm{em}}_{ab}$ where

$$T^{\mathrm{f}}_{ab} = (\rho + p) U_a U_b - \epsilon\, p g_{ab} \tag{17}$$

is the energy-momentum tensor for an *ideal fluid*, while $\rho$ and $p$ are, respectively, the *total energy density* and *pressure* as measured by an observer moving with the fluid. The time-like vector field (*flow vector*) $U^a$ denotes the normalized future directed 4-velocity of the fluid. It satisfies

$$U^a U_a = \epsilon. \tag{18}$$

Associated to $U^a$ we introduce the *projection tensor*

$$h_{ab} \equiv g_{ab} - \epsilon U_a U_b, \tag{19}$$

projecting onto the three dimensional subspace orthogonal to $U^a$. Indices are raised and lowered with $g_{ab}$. Thus, one has that $h^a{}_b = \delta^a{}_b - \epsilon U^a U_b$, $h^b{}_a h^a{}_c = h^b{}_c$, and $h^a{}_b U_a = 0$.

### 3.1 The electromagnetic energy-momentum tensor

The tensor $T^{\mathrm{em}}_{ab}$ denotes the energy momentum tensor of an electromagnetic field:

$$T^{\mathrm{em}}_{ab} = -\epsilon \left( F_{ac} F_b{}^c - \frac{1}{4} F_{cd} F^{cd} g_{ab} \right), \tag{20}$$

where $F_{ab}$ is the *electromagnetic field (Faraday) tensor*. The latter can be split in its *electric part*, $E_a \equiv F_{ab} U^b$, and its *magnetic part*, $B^a \equiv \frac{1}{2} \epsilon^{abcd} U_b F_{cd}$, with respect to the flow. Indeed, one has that

$$F_{ab} = \epsilon (2 E_{[a} U_{b]} - \epsilon_{abcd} B^c U^d). \tag{21}$$

Alternatively, the one can write

$$F_{ab} = \epsilon \left( E_{ab} - {}^*B_{ab} \right), \tag{22}$$

where

$$E_{ab} \equiv 2 E_{[a} U_{b]}, \quad {}^*E_{ab} \equiv \epsilon_{abcd} E^c U^d, \quad B_{ab} \equiv 2 B_{[a} U_{b]}, \quad {}^*B_{ab} \equiv \epsilon_{abcd} B^c U^d. \tag{23}$$

One can readily verify the properties

$$B_{ab}U^b = \epsilon B_a, \quad E_{ab}U^b = \epsilon E_a, \quad {}^*B_{ab}U^b = {}^*E_{ab}U^b = 0, \tag{24}$$

where, as before, $^*$ denotes the Hodge dual operator. In particular, one has that,$^{**}B_{ab} = -B_{ab}$, so that ${}^*F_{ab} = \epsilon\left({}^*E_{ab} - {}^{**}B_{ab}\right) = \epsilon\left({}^*E_{ab} + B_{ab}\right)$, with

$$B_a = {}^*F_{ab}U^b, \quad {}^*F_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}, \quad {}^{**}F_{ab} = -F_{ab}. \tag{25}$$

From the decomposition into electric and magnetic parts, one can readily verify that the Faraday tensor has six independent components as these vectors are both spatial —that is, $E_aU^a = B_aU^a = 0$. Using the decomposition into electric and magnetic parts, the electromagnetic energy-momentum tensor, Eq. (20), can be written as

$$T^{\mathrm{em}}_{ab} \equiv -\frac{\epsilon}{2}U_aU_b(E^2+B^2) + \frac{h_{ab}}{6}(E^2+B^2) + P_{ab} - 2\epsilon\mathcal{G}_{(a}U_{b)}, \tag{26}$$

where we have written $E^2 \equiv E_aE^a$ and $B^2 \equiv B^aB_a$, and $P_{ab}$ denotes the symmetric, trace-free tensor given by

$$P_{ab} = P_{(ab)} \equiv \frac{h_{ab}}{3}(E^2+B^2) - (E_aE_b + B_aB_b). \tag{27}$$

Furthermore,

$$\mathcal{G}_a \equiv \epsilon_{auvd}E^uB^vU^d, \tag{28}$$

denotes the *Poynting vector*. Alternatively, one can rewrite Eq. (26) in the form

$$T^{\mathrm{em}}_{ab} = \frac{g_{ab}}{2}(E^2+B^2) - (E_aE_b + B_aB_b) - 2\epsilon\mathcal{G}_{(a}U_{b)} - \epsilon U_aU_b(E^2+B^2). \tag{29}$$

### 3.2 The Maxwell equations

The Maxwell equations are given by

$$\nabla_{[a}F_{bc]} = 0, \quad \nabla^aF_{ab} = \epsilon J_b. \tag{30}$$

Using Eq. (22) one can rewrite them as

$$\nabla_{[a}\left(E_{bc]} - {}^*B_{bc]}\right) = 0, \quad \nabla^a\left(E_{ab} - {}^*B_{ab}\right) = J_b. \tag{31}$$

The homogeneous Maxwell equation can be rewritten in terms of the Hodge dual of $F_{ab}$ as

$$\nabla^{a*}F_{ab} = \epsilon\nabla^a\left({}^*E_{ab} + B_{ab}\right) = 0. \tag{32}$$

Eq. (31) can be further expanded to yield

$$\left(\nabla^bE_b\right)U_a - \dot{E}_a + \nabla^b\epsilon_{abcd}B^cU^d + 2E_{[b}\nabla^bU_{a]} = J_a, \tag{33}$$

where $\dot{E} \equiv U^a \nabla_a E_b$ stands for the covariant derivative of $E_a$ along the flow. For Eq. (32) a similar computation gives

$$\nabla^a \left( \epsilon_{abcd} E^c U^d + 2B_{[a} U_{b]} \right) = 0. \tag{34}$$

Projecting Eq. (33) along the directions parallel and orthogonal to the flow vector $U^b$ one obtains the two equations

$$\epsilon \nabla^a E_a + E_b \dot{U}^b - \epsilon_{abcd} U^a B^b \nabla^c U^d = U^b J_b, \tag{35}$$

$$\begin{aligned} &\epsilon U_e \epsilon_{abcd} U^b B^c \nabla^a U^d - \left( \dot{E}_e + \epsilon U_e E^b \dot{U}_b \right) - \nabla^a (\epsilon_{aecd} B^c U^d) \\ &+ h^b{}_e E_a \nabla^a U_b - E_e \nabla^a U_a = h^b{}_e J_b, \end{aligned} \tag{36}$$

Projecting Eq. (34) along the directions parallel and orthogonal to $U^a$ one obtains

$$\epsilon_{abcd} U^b E^c \nabla^a U^d + \epsilon \nabla^a B_a - U^b \dot{B}_b = 0, \tag{37}$$

$$\epsilon_{abcd} h^b{}_f \nabla^a (E^c U^d) + B_a h^b{}_f \nabla^a U_b - h^b{}_f \dot{B}_b - B_f \nabla^a U_a = 0, \tag{38}$$

where it has been used that $\nabla_a (E^b U_b) = E_b \nabla_a U^b + U_b \nabla_a E^b = 0$.

The electromagnetic current vector $J^a$ will be split with respect to the flow vector as

$$J^a = \rho_{\mathrm{C}} U^a + j^a, \tag{39}$$

where $\rho_{\mathrm{C}}$ denotes the *charge density* and $j^a$ is the *orthogonally projected conduction current*. Using the decomposition Eq. (39) in Eqs.(35) and (36), one obtains the following propagation equations for the electric and magnetic parts of the Faraday tensor:

$$\dot{E}_{\langle f \rangle} = -2E^a h_{f[a} \nabla_{b]} U^b - \epsilon_{abcd} h^b{}_f \nabla^a (B^c U^d) - j_f, \tag{40}$$

$$\dot{B}_{\langle f \rangle} = -2B^a h_{f[a} \nabla_{b]} U^b + \epsilon_{abcd} h^b{}_f \nabla^a (E^c U^d), \tag{41}$$

and from Eqs.(37) and (38) the constraint equations

$$\epsilon D^a B_a = -\epsilon_{abcd} U^b E^c \nabla^a U^d, \tag{42}$$

$$\epsilon D^a E_a = \epsilon_{abcd} U^a B^b \nabla^c U^d + \epsilon \rho_{\mathrm{C}}, \tag{43}$$

where $\dot{w}_{\langle a \rangle} \equiv h_a{}^b \dot{w}_b$ denotes the directional covariant derivative along the flow vector (*Fermi derivative*) and $D_a w_b \equiv h_a{}^u h_b{}^v \nabla_u w_v$ is the orthogonally projected covariant derivative of a vector. We note that $\epsilon_{abcd} h_f{}^b \nabla^a (X^c U^d) = -\mathrm{curl} X_f + \epsilon \epsilon_{afcd} U^a X^c \dot{U}^d$, where $\mathrm{curl} X_f \equiv \epsilon_{facd} U^d \nabla^a B^c$.

In the sequel, it will be necessary to specify the form of the conduction current, $j^a$. Consistent with *Ohm's law*, we assume a linear relation between the conduction current $j^a$ and the electric field. More precisely, we set

$$j^a = \sigma^{ab} E_b, \tag{44}$$

where $\sigma^{ab}$ denotes the *conductivity* of the fluid (plasma). We will restrict our attention to isotropic fluids for which $\sigma^{ab} = \sigma g^{ab}$, so that Eq. (39) becomes

$$J^a = \rho_{\rm C} U^a + \sigma E^a, \tag{45}$$

with $\sigma$ the *electrical conductivity coefficient*. Ideal MHD is characterised by the condition, $\sigma \to \infty$ (ideal conductive plasma). It follows from Eq. (45) that for this to be the case one requires the constraint $E_a = 0$.

### 3.3 The fluid equations

Form the conservation of the energy-momentum tensor

$$\nabla^a T_{ab} = 0, \tag{46}$$

and using Eqs. (17) and (20) and the Maxwell equation Eq. (30) one obtains

$$\nabla^a T^{\rm em}_{ab} = -\epsilon \left(\nabla^a F_{ac}\right) F_b{}^c, \tag{47}$$

$$\nabla^a T^{\rm f}_{ab} = U_b U_a \nabla^a (p+\rho) + (p+\rho) \left[U_b (\nabla^a U_a) + U_a \nabla^a U_b\right] - \epsilon \nabla_b p. \tag{48}$$

Therefore, Eq. (46) reads:

$$U_b U_a \nabla^a (p+\rho) + (p+\rho) \left[U_b (\nabla^a U_a) + U_a \nabla^a U_b\right] - \epsilon \nabla_b p - \epsilon \left(\nabla^a F_{ac}\right) F_b{}^c = 0. \tag{49}$$

In what follows, we will consider the projections of (49) along the directions parallel and orthogonal to the flow lines of the fluid. Contracting Eq. (49) with $\epsilon U^b$ we obtain the conservation equation

$$U_a \nabla^a \rho + (p+\rho) \nabla^a U_a - U^b F_b{}^c (\nabla^a F_{ac}) = 0, \tag{50}$$

where the constraint (18) has been used. For an ideally conducting fluid, where $E_a = F_{ab} U^b = 0$, the last term of Eq. (50) is identically zero and the electromagnetic field does not have a direct effect on the conservation equation along the flow lines. Contracting (49) with the projector $h^{bc}$ one obtains the *Euler equation*

$$(p+\rho) U^a \nabla_a U^c - \epsilon h^{bc} \nabla_b p - \epsilon (\nabla^a F_{ad}) F_b{}^d h^{bc} = 0. \tag{51}$$

The later equation can also be written as

$$(p+\rho) U_a \nabla^a U_b + (U_b U^d \nabla_d - \epsilon \nabla_b) p - \epsilon (\nabla^a F_{ad}) \left(F_b{}^d - \epsilon F^{ed} U_e U_b\right) = 0. \tag{52}$$

In the ideal MHD case the last term of Eq. (52) is identically zero so that Eq. (51) reduces to

$$(p+\rho) U^a \nabla_a U^c - \epsilon h^{bc} \nabla_b p - \epsilon (\nabla^a F_{ad}) F^{cd} = 0. \tag{53}$$

### 3.4 Thermodynamic considerations

Here we consider a *one species particle fluid* (simple fluid). We denote by $n$, $s$, $T$ the *particle number density*, the *entropy per particle* and the *absolute temperature*, as measured by comoving observers. We also introduce the *volume per particle*, $v$, and the *energy per particle*, $e$, via

$$v \equiv \frac{1}{n}, \quad e \equiv \frac{\rho}{n}. \tag{54}$$

In terms of these variables the first law of Thermodynamics, $\mathrm{d}e = -p\mathrm{d}v + T\mathrm{d}s$, takes the form

$$\mathrm{d}\rho = \frac{p+\rho}{n}\mathrm{d}n + nT\mathrm{d}s. \tag{55}$$

Assuming an equation of state of the form $\rho = f(n,s) \geq 0$, one obtains from Eq. (55) that

$$p(n,s) = n\left(\frac{\partial\rho}{\partial n}\right)_s - \rho(n,s), \quad T(n,s) = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_n. \tag{56}$$

Assuming that $\partial p/\partial\rho > 0$ we define the *speed of sound*, $\nu_s = \nu_s(n,s)$, by

$$\nu_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{n}{\rho+p}\frac{\partial p}{\partial n} > 0. \tag{57}$$

Since we are not considering particle annihilation or creation we consider the equation of conservation of particle number:

$$U^a\nabla_a n + n\nabla_a U^a = 0. \tag{58}$$

Combining this equation with Eqs. (50) and (55) we obtain

$$U^a\nabla_a s = \frac{1}{nT}U^b F_b{}^c \nabla^a F_{ac}. \tag{59}$$

In the case of an infinitely conducting plasma, where the last term of Eq. (55) vanishes, Eq. (59) describes an adiabatic flow —that is, $U^a\nabla_a s = 0$, so that the entropy per particle is conserved along the flow lines. A particular case of interest is when $s$ is a constant of both space and time. In this case the equation of state can be given in the form $p = p(\rho)$. A further discussion on homentropic flows and barotropic equations of state is provided in Appendix A.

## 4 The hyperbolic reduction procedure

### 4.1 General considerations

Following [3], we introduce the notation

$$N^a \equiv \delta_0{}^a, \quad N \equiv N^a e_a = e_0. \tag{60}$$

For a given tensor any contraction with $N$ will be denoted by replacing the corresponding index by $N$, and the projection with respect to $h_a{}^b$ will be indicated by a prime. Accordingly, for a tensor $T_{abc}$ one writes

$$T'_{aNb} \equiv T_{mpq} h_a{}^m N^p h_b{}^q. \tag{61}$$

Introducing the notation $\epsilon_{abc} = \epsilon'_{Nabc}$, where $\epsilon_{0123} = 1$, we obtain the decomposition

$$\epsilon_{abcd} = 2\epsilon \left( N_{[a}\epsilon_{b]cd} - \epsilon_{ab[c}N_{d]} \right). \tag{62}$$

Given a spatial tensor $T_{a_1\cdots a_p} = T'_{a_1\cdots a_p}$ we define the *spatial covariant derivative*

$$\mathcal{D}_a T_{a_1\cdots a_p} = h_a{}^b h_{a_1}{}^{b_1} \cdots h_{a_p}{}^{b_p} \nabla_b T_{b_1\cdots b_p}. \tag{63}$$

One has, in particular, that $\mathcal{D}_a h_{bc} = \mathcal{D}_a \epsilon_{bcd} = 0$.

For convenience, we introduce the tensors

$$a^a \equiv N^b \nabla_b N^a, \quad \chi_{ab} \equiv h_a^c \nabla_c N_b, \quad \chi \equiv h^{ab}\chi_{ab}. \tag{64}$$

In terms of these we can write

$$\nabla_a N^b = \epsilon N_a a^b + \chi_a{}^b, \quad a^a = h_b{}^a \Gamma^b_{00}, \quad \chi_{ab} = -\epsilon h_a{}^c h_b{}^d \Gamma^0{}_{cd}. \tag{65}$$

Key for our subsequent discussion will be the decomposition of the Weyl tensor on terms of its *electric*, $\hat{E}_{ab} \equiv C'_{NaNb}$, and *magnetic* parts, $\hat{B}_{ab} = C^{*\prime}{}_{NaNb}$, with respect to $N_a$ given by

$$C_{abcd} = 2\epsilon(l_{b[c}\hat{E}_{d]a} - l_{a[c}\hat{E}_{d]b}) - 2(N_{[c}\hat{B}_{d]p}\epsilon^p{}_{ab} + N_{[a}\hat{B}_{b]p}\epsilon^p{}_{cd}) \tag{66}$$

where $l_{ab} \equiv h_{ab} - \epsilon N_a N_b$. From the Bianchi identity Eq. (13) for the tensor $F_{abcd}$ we obtain the decomposition

$$F_{abc} = N_a \left( F'_{NbN}N_c - F'_{NcN}N_b \right) - 2\epsilon F'_{aN[b}N_{c]} + \epsilon N_a F'_{Nbc} + F'_{abc}, \tag{67}$$

from where it follows that

$$\begin{aligned}\epsilon F'_{aNb} = \mathcal{L}_N F'_{NaNb} + \epsilon \mathcal{D}^c F'_{caNb} - a^c \left( F'_{Nacb} + F'_{caNb} \right) + \epsilon a_a F'_{NNNb} \\ -\epsilon \chi^{cd} F'_{cadb} - \chi_a{}^c F'_{NcNb} - \chi_b{}^c F'_{NaNc} + \chi^c{}_a F'_{cNNb} + \chi F'_{NaNb}\end{aligned} \tag{68}$$

4.2 Detailed analysis of the evolution equations

In the sequel we make the following specific choice of timelike frame vector:

$$N = e_0 = U, \quad N_a = U_a.$$

Using the decomposition given by Eq. (14) we compute the following components of the tensors $F_{abcd}$ and $\tilde{F}_{abcd}$:

$$\tilde{F}'_{NNNa} = 0, \quad F'_{NNNa} = -\frac{\epsilon\kappa}{2} T^{\rm em}_{ec} U^c h^e{}_a, \tag{69}$$

$$\tilde{F}'_{NaNb} = \hat{B}_{ab}, \qquad F'_{NaNb} = \hat{E}_{ab} + \frac{\kappa\rho}{6} h_{ab} - \frac{\kappa\epsilon}{2} T^{\rm em}_{dc} h^d{}_b h^c{}_a, \tag{70}$$

$$\tilde{F}'_{NNab} = \frac{\kappa}{2}\left(T^{\rm em}_{pv} U^v\right)\epsilon^p_{uab} U^u, \quad F'_{NNab} = 0, \tag{71}$$

$$\tilde{F}'_{aNNb} = -\hat{B}_{ab} + \frac{\kappa}{2}\left(T^{\rm em}_{pv} U^v\right)\epsilon^p_{aub} U^u, \tag{72}$$

$$F'_{aNNb} = -\hat{E}_{ab} + \frac{\kappa}{2} h_{ab}\left[\left(\frac{2}{3}\rho + p\right) + T^{\rm em}_{uv} U^v U^u\right], \tag{73}$$

$$\tilde{F}'_{Nabc} = \epsilon\hat{E}_{ap}\epsilon^p_{bc} - \frac{\kappa\rho\epsilon}{6}\epsilon_{aubc} U^u + \frac{\kappa}{2}\left(T^{\rm em}_{pv} h^v{}_a\right)\epsilon^p{}_{ubc} U^u, \quad F'_{Nabc} = -\epsilon\hat{B}_{ap}\epsilon^p{}_{bc}, \tag{74}$$

$$\tilde{F}'_{aNbc} = -\epsilon\hat{E}_{ap}\epsilon^p{}_{bc} + \frac{\epsilon\kappa}{2}\left(\frac{2}{3}\rho + p\right) U_p \epsilon^p{}_{abc} + \frac{\kappa}{2}\left(T^{\rm em}_{up} U^u\right)\epsilon^p{}_{dvf} h^d{}_a h^v{}_b h^f{}_c, \tag{75}$$

$$F'_{aNbc} = \hat{B}_{ap}\epsilon^p{}_{bc} - \frac{\kappa}{2}\left[h_{ab}\left(T^{\rm em}_{fu} U^u h^f{}_c\right) - h_{ac}\left(T^{\rm em}_{fu} U^u h^f{}_b\right)\right], \tag{76}$$

$$\tilde{F}'_{abNc} = -2\epsilon\hat{E}_{p[b}\epsilon_{a]}{}^p{}_c - \frac{\kappa\rho\epsilon}{6}\epsilon_{bavc} U^v + \frac{\kappa}{2} T^{\rm em}_{pu} h^u{}_b \epsilon^p{}_{avc} U^v, \tag{77}$$

$$F'_{abNc} = -\epsilon\hat{B}_{cp}\epsilon^p{}_{ab} + \frac{\kappa}{2} h_{ac} T^{\rm em}_{uv} U^u h^v{}_b, \tag{78}$$

$$\tilde{F}'_{abcd} = -\hat{B}_{pq}\epsilon^p{}_{ab}\epsilon^q{}_{cd} - \frac{\epsilon\rho\kappa}{6}\epsilon_{puef} h^p{}_b h^u{}_a h^e{}_c h^f{}_d + \frac{\kappa}{2}\left(T^{\rm em}_{pv} h^v{}_b\right)\epsilon^p{}_{uef} h^u{}_a h^e{}_c h^f{}_d, \tag{79}$$

$$F'_{abcd} = 2\epsilon\left[l_{b[c}\hat{E}_{d]a} - l_{a[c}\hat{E}_{d]b}\right] + \frac{\epsilon\kappa\rho}{3} h_{c[a} h_{b]d} - \kappa T^{\rm em}_{uv} h^v{}_b h_{c[a} h_{d]}{}^u. \tag{80}$$

Moreover, Eq. (9) becomes,

$$S_{ab} = S^{\rm f}_{ab} + S^{\rm em}_{ab}, \tag{81}$$

where

$$S^{\rm f}_{ab} \equiv \kappa\left[T^{\rm f}_{ab} - \frac{1}{3} g_{ab} g^{cd} T^{\rm f}_{cd}\right] = \kappa\left[\left(\frac{2}{3}\rho + p\right) U_a U_b - \frac{\epsilon\rho}{3} h_{ab}\right], \tag{82}$$

and

$$S^{\rm em}_{ab} \equiv \kappa T^{\rm em}_{ab} = -\epsilon\kappa\left(F_{ac} F^{dc} g_{db} - \frac{1}{4} F_{cd} F^{cd} g_{ab}\right). \tag{83}$$

Using Eq. (81) and Eq. (66) in Eq. (8), the Riemann curvature tensor $R_{abcd}$, can be decomposed in the form

$$R_{abcd} = R^{\rm w}_{abcd} + R^{\rm m}_{abcd} + R^{\rm em}_{abcd}, \tag{84}$$

where

$$R^{\rm em}_{abcd} \equiv \kappa\Bigg((E^2+B^2)\left[\frac{1}{2}\left(g_{a[c}g_{d]b}-g_{b[c}g_{d]a}\right)-2\epsilon U_{[b}g_{a][c}U_{d]}\right]-2E_{[b}g_{a][c}E_{d]}+2B_{[b}g_{a][c}B_{d]}$$
$$+2\epsilon\left[U_{[a}g_{b][c}\mathcal{G}_{d]}-\mathcal{G}_{[a}g_{b][c}U_{d]}\right]\Bigg), \tag{85}$$

and

$$R^{\rm W}_{abcd} \equiv 2\epsilon(l_{b[c}\hat{E}_{d]a}-l_{a[c}\hat{E}_{d]b})-2(U_{[c}\hat{B}_{d]p}\epsilon^{p}{}_{ab}+U_{[a}\hat{B}_{b]p}\epsilon^{p}{}_{cd}), \tag{86}$$
$$R^{\rm m}_{abcd} = \frac{1}{3}\epsilon\kappa\rho\left(l_{b[c}h_{d]a}-l_{a[c}h_{d]b}\right)+\kappa p\left(h_{a[c}U_{d]}U_b-h_{b[c}U_{d]}U_a\right). \tag{87}$$

Finally, from the decomposition Eq. (68) we identify the tensors

$$\epsilon\tilde{F}'_{(a|N|b)}, \qquad \epsilon\left[F'_{(a|N|b)}-\frac{1}{2}h_{ab}h^{uv}F'_{uNv}\right],$$

As it will be shown in the next subsections, the evolution equations for the electric and magnetic parts of the Weyl tensor will be extracted from these tensors. The essential difficulty in our analysis resides in a satisfactory treatment of these equations.

#### 4.2.1 Evolution equation for $\hat{B}_{ab}$

From Eqs. (68) and (13) we obtain the following evolution equation for the magnetic part of the Weyl tensor

$$0=\epsilon\tilde{F}'_{(a|N|b)}=\mathcal{L}_U\hat{B}_{ab}-D_d\hat{E}_{c(a}\epsilon_{b)}{}^{dc}+2\epsilon a_c\epsilon^{cd}{}_{(a}\hat{E}_{b)d}-\chi^{c}{}_{(a}\hat{B}_{b)c}$$
$$-2\chi_{(a}{}^{c}\hat{B}_{b)c}+\chi\hat{B}_{ab}-\epsilon\chi_{cd}\hat{B}_{pq}\epsilon^{pc}{}_{(a}\epsilon^{dq}{}_{b)}+\epsilon\tilde{F}'^{\rm em}_{(a|N|b)}, \tag{88}$$

where

$$\epsilon\tilde{F}'^{\rm em}_{(a|N|b)} \equiv \frac{\kappa\epsilon}{2}\mathcal{D}^c\left(T^{\rm em}_{up}\epsilon^{p}{}_{cv(b}h^{u}_{a)}U^v\right)-\frac{\epsilon\kappa}{2}\chi^{cd}\left(T^{\rm em}_{pv}h^{v}{}_{(a}h^{f}{}_{b)}\right)\epsilon^{p}{}_{uef}h^{u}{}_{c}h^{e}{}_{d}+\frac{\kappa}{2}\epsilon^{p}{}_{cu(b}\chi^{c}{}_{a)}U^u\left(T^{\rm em}_{pv}U^v\right). \tag{89}$$

In vacuum, Eq. (88) together with its electric counterpart provide hyperbolic evolution equations for the electric and magnetic parts of the Weyl tensor independently of the gauge choice. This is not automatically the case in the presence of matter. Using Eq. (83) in Eq. (89) we find that the electromagnetic contribution to the evolution equation of $\hat{B}_{ab}$ is given by

$$\epsilon\tilde{F}'^{\rm em}_{(a|N|b)} \equiv \frac{-\epsilon\kappa}{2}\Bigg(\epsilon\mathcal{D}^c\left[U^v\epsilon^{p}{}_{cv(b}\left(h^{u}{}_{a)}F_{uq}F_{p}{}^{q}-\frac{1}{4}h_{a)p}F_{qs}F^{qs}\right)\right]$$
$$-\epsilon\chi^{ue}\epsilon^{p}{}_{uef}\left(h^{f}{}_{(b}h^{v}{}_{a)}F_{vc}F_{p}{}^{c}-\frac{1}{4}h^{f}{}_{(b}h_{a)p}F_{qs}F^{qs}\right)-U^u\epsilon^{p}{}_{cu(b}\chi^{c}{}_{a)}E_dF_p{}^{d}\Bigg) \tag{90}$$

Notice that this last expression contains derivatives of the Faraday tensor which cannot be replaced by means of the Maxwell equations. These derivatives enter into the principal part of the evolution equations and destroy the

hyperbolicity of the evolution equations for the magnetic part of the Weyl tensor. In order to deal with this difficulty and additional variable, representing the derivative of the Faraday tensor needs to be introduced. This will be discussed in subsection 4.3.

*4.2.2 Evolution equation for $\hat{E}_{ab}$*

The evolution equation for the electric part of the Weyl tensor can be obtained from Eq. (68) as follows:

$$0 = \epsilon \left[ F'_{(a|N|b)} - \frac{1}{2} h_{ab} h^{uv} F'_{uNv} \right] = \mathcal{L}_U \hat{E}_{ab} + D_c \hat{B}_{d(a} \epsilon_{b)}{}^{cd} - 2\epsilon a_c \epsilon^{cd}{}_{(a} \hat{B}_{b)d} - 3\chi_{(a}{}^{c} \hat{E}_{b)c} - 2\chi^{c}{}_{(a} \hat{E}_{b)c}$$
$$+ h_{ab} \chi^{cd} \hat{E}_{cd} + 2\chi \hat{E}_{ab} + \frac{\kappa}{2}(\rho + p) \left[ \chi_{(ab)} - \frac{1}{3} \chi h_{ab} \right] + \epsilon \left[ F'^{\text{em}}_{(a|N|b)} - \frac{1}{2} h_{ab} h^{uv} F'^{\text{em}}_{uNv} \right], \tag{91}$$

where

$$\epsilon \left[ F'^{\text{em}}_{(a|N|b)} - \frac{1}{2} h_{ab} h^{uv} F'^{\text{em}}_{uNv} \right] = \frac{1}{4} \epsilon \kappa \mathcal{L}_U \left( T^{\text{em}}_{dc} h^{d}{}_{a} h^{c}{}_{b} \right) + \frac{\epsilon\kappa}{2} \left[ h^{v}{}_{(a} \mathcal{D}_{b)} - \frac{1}{2} h_{ab} \mathcal{D}^{v} \right] \left( T^{\text{em}}_{uv} U^{u} \right)$$
$$+ \kappa T^{\text{em}}_{uv} U^{v} \left[ \frac{h_{ab}}{2} a^{u} - h^{u}{}_{(b} a_{a)} \right] + \frac{\kappa}{2} T^{\text{em}}_{uv} U^{u} U^{v} \left[ \chi_{(ab)} - \frac{h_{ab}}{2} \chi \right]$$
$$+ \frac{\kappa\epsilon}{2} \left[ 2\chi_{(a}{}^{u} h^{v}{}_{b)} - \chi_{(a}{}^{u} h_{b)}{}^{v} - \frac{\chi^{uv} h_{ab}}{2} \right] T^{\text{em}}_{uv}. \tag{92}$$

Finally, using Eq. (83) we obtain the explicit expression

$$\epsilon \left[ F'^{\text{em}}_{(a|N|b)} - \frac{h_{ab} h^{uv}}{2} F'^{\text{em}}_{uNv} \right] = -\frac{\kappa \mathcal{L}_U}{4} \left[ h^{q}{}_{(a} h^{c}{}_{b)} F_{qf} F_{c}{}^{f} - \frac{h_{ab} F_{qp} F^{qp}}{2} \right] - \frac{\kappa}{2} \left[ \frac{h_{ab} \mathcal{D}^{v}}{2} - h^{v}{}_{(a} \mathcal{D}_{b)} \right] \left( E^{c} F_{vc} \right)$$
$$- \kappa \epsilon E^{c} F_{vc} \left( h^{v}{}_{(a} a_{b)} - \frac{h_{ab}}{2} a^{v} \right) - \frac{\kappa}{2} F_{ud} F_{v}{}^{d} \left[ 2\chi_{(a}{}^{v} h_{b)}{}^{u} - \chi^{v}{}_{(a} h_{b)}{}^{u} - \frac{1}{2} \chi^{uv} h_{ab} \right]$$
$$+ \frac{\kappa}{4} \chi_{(ab)} F_{qp} F^{qp} - \frac{\kappa\epsilon}{2} E^{2} \chi_{(ab)} - \frac{1}{8} \chi h_{ab} F_{qp} F^{qp} + \frac{1}{4} \epsilon \kappa \chi h_{ab} E^{2}. \tag{93}$$

As in the previous subsection we observe the presence of derivatives of the Faraday tensor which need to be dealt with by the introduction of a new field if one is to preserve the hyperbolicity of the equations.

## 4.3 An auxiliary field

As discussed previously, derivatives of the Faraday tensor appear in the evolution equations implied by the Bianchi identity. This feature of non-vacuum systems destroys, in principle, the hyperbolicity of the evolution equations. In order to get around with this difficulty, it is necessary to introduce the covariant derivative of the Faraday tensor as a field variable. More precisely, let

$$\psi_{abc} \equiv \nabla_a F_{bc}. \tag{94}$$

As a consequence of the definition and the Maxwell equations the tensor $\psi_{abc}$ has the symmetries $\psi_{abc} = \psi_{a[bc]}$, $\psi_{[abc]} = 0$. In order to construct a suitable equation for $\psi_{abc}$ we consider the commutator of the covariant derivative $\nabla$ in the form

$$\nabla_a \nabla_b F_{cd} - \nabla_b \nabla_a F_{cd} = -2F_{e[c} R^e{}_{d]ab}. \tag{95}$$

Making use of the definition of $\psi_{abc}$ one obtains

$$\nabla_a \psi_{bcd} - \nabla_b \psi_{acd} = -2F_{e[c} R^e{}_{d]ab}, \tag{96}$$

where in the right hand side we use the expression for $R_{abcd}$ involving the its decomposition in terms of the Weyl tensor and the matter fields. The tensorial expression (96) provides the required field equations.

### 4.4 Propagation equations for the auxiliary field

In order to deduce from them suitable evolution equations one proceeds by analogy to the case of the Faraday tensor $F_{ab}$. Proceeding as in the case of Eqs. (30) and (32) we obtain the following two equations for $\psi_{abc}$

$$\nabla^b \psi_{adb} = 2F^{e[b} R_{d]aeb} - \epsilon \nabla_a J_d, \tag{97}$$

$$\nabla^{a*} \psi_{cab} = \epsilon_b{}^{aud} F_{eu} R^e{}_{dac}, \tag{98}$$

where $R_{abcd}$ is given by Eq. (84). In the subsequent discussion is convenient to introduce here the following notation

$$\psi_{ayz} \equiv \Psi_{yz} = \Psi_{[yz]}. \tag{99}$$

That is, the first index of $\psi_{abc}$ tensor is dropped. Because of the $\psi_{abc}$ is antisymmetric with respect to the last pair of indices, it is natural to introduce its *electric* and *magnetic* part respect to $U_a$ via

$$\mathcal{E}_d \equiv \Psi_{dN} \equiv \psi_{adn} U^n, \qquad \mathcal{B}^u \equiv \frac{1}{2} \epsilon^{uvzt} U_v \Psi_{zt}, \tag{100}$$

respectively, so that are $\mathcal{E}_d$ and $\mathcal{B}_d$ are spacelike vectors:

$$\mathcal{E}_a U^a = \mathcal{B}_a U^a = 0. \tag{101}$$

Accordingly, we write the tensor $\Psi$ as

$$\Psi_{ab} = \epsilon \left[\mathcal{E}_{ab} - {}^*\mathcal{B}_{ab}\right]. \tag{102}$$

Furthermore, one has that

$${}^*\Psi_{ab} = \epsilon \left[{}^*\mathcal{E}_{ab} + \mathcal{B}_{ab}\right], \tag{103}$$

where

$$\mathcal{E}_{ab} \equiv 2\mathcal{E}_{[a} U_{b]}, \qquad {}^*\mathcal{E}_{ab} \equiv \epsilon_{abcd} \mathcal{E}^c U^d, \tag{104}$$

$$\mathcal{B}_{ab} \equiv 2\mathcal{B}_{[a} U_{b]}, \qquad {}^*\mathcal{B}_{ab} \equiv \epsilon_{abcd} \mathcal{B}^c U^d. \tag{105}$$

Continuing the analogy with the Maxwell equations and the Faraday tensor $F_{ab}$, it follows that Eqs. (97) and Eq. (98) read

$$\dot{\mathcal{E}}_d - \left(\nabla^b \mathcal{E}_b\right) U_d - \nabla^b \epsilon_{dbcv} \mathcal{B}^c U^v - 2\mathcal{E}_{[b} \nabla^b U_{d]} = \epsilon \mathcal{S}_d, \tag{106}$$

$$\nabla^a \left(\epsilon_{abcd} \mathcal{E}^c U^d + 2\mathcal{B}_{[a} U_{b]}\right) = \epsilon \mathcal{V}_b, \tag{107}$$

where

$$\mathcal{S}_d \equiv 2F^{e[b} R_{d]feb} - \epsilon \nabla_f J_d, \quad \mathcal{V}_b \equiv -\epsilon_f{}^{aud} F_{eu} R^e{}_{dab}. \tag{108}$$

Projecting Eq. (106) along the directions parallel and orthogonal to the 4-velocity $U^b$, one obtains the following set of two equations:

$$\mathcal{E}_b \dot{U}^b + \epsilon \nabla^a \mathcal{E}_a - \epsilon_{abcd} U^a \mathcal{B}^b \nabla^c U^d = -\epsilon U^b \mathcal{S}_b, \tag{109}$$

$$\dot{\mathcal{E}}_e + \epsilon U_e \mathcal{E}^b \dot{U}_b + \mathcal{E}_e \nabla^a U_a - h^b{}_e \mathcal{E}_a \nabla^a U_b - \epsilon U_e \epsilon_{abcd} U^b \mathcal{B}^c \nabla^a U^d + \nabla^a (\epsilon_{aecd} \mathcal{B}^c U^d) = -\epsilon h^b{}_e \mathcal{S}_b. \tag{110}$$

A similar procedure applied to Eq. (107) gives

$$\epsilon_{abcd} U^b \mathcal{E}^c \nabla^a U^d + \epsilon \nabla^a \mathcal{B}_a + B^b \dot{\mathcal{U}}_b = \epsilon U^b \mathcal{V}_b, \tag{111}$$

$$\epsilon_{abcd} h^b{}_f \nabla^a (\mathcal{E}^c U^d) + \mathcal{B}_a h^b{}_f \nabla^a U_b - h^b{}_f \dot{\mathcal{B}}_b - \mathcal{B}_f \nabla^a U_a = \epsilon h_f{}^b \mathcal{V}_b. \tag{112}$$

The propagation equations (109)-(110) can be written as

$$\dot{\mathcal{E}}_{\langle f \rangle} = -2\mathcal{E}^a h_{f[a} \nabla_{b]} U^b - \epsilon_{abcd} h^b{}_f \nabla^a (\mathcal{B}^c U^d) + \epsilon h^d{}_f \mathcal{S}_d, \tag{113}$$

$$\dot{\mathcal{B}}_{\langle f \rangle} = -2\mathcal{B}^a h_{f[a} \nabla_{b]} U^b + \epsilon_{abcd} h^b{}_f \nabla^a (\mathcal{E}^c U^d) - \epsilon h^d{}_f \mathcal{V}_d. \tag{114}$$

while the constraint equations (111)-(112) assume the form

$$\epsilon D^a \mathcal{B}_a = -\epsilon_{abcd} U^b \mathcal{E}^c \nabla^a U^d + \epsilon U^d \mathcal{V}_d, \tag{115}$$

$$\epsilon D^a \mathcal{E}_a = \epsilon_{abcd} U^a \mathcal{B}^b \nabla^c U^d - \epsilon U^d \mathcal{S}_d. \tag{116}$$

In order to verify that Eqs. (113)-(114) give appropriate hyperbolic evolution equations, we need the explicit expression for the electromagnetic current vector $J^a$ given by Ohm's law —see Eq. (45). A computation then gives that

$$\mathcal{V}_b h^b{}_v = -\epsilon_f{}^{aud} F_{eu} R^e{}_{dab} h^b_v. \tag{117}$$

$$\mathcal{S}_d h^d{}_v = F^{eb} R_{dfeb} h^d{}_v + R_{fe} \left(F^e_{\ v} - \epsilon E^e U_v\right) - \epsilon \left(\nabla_f (\sigma E_v) + \epsilon U_v \sigma E_d \nabla_f U^d + \rho_C \nabla_f U_v\right).$$

In the last equation it is assumed that the derivatives of the components of the Faraday tensor appearing in the right hand side of (118) are rewritten in terms of the tensor $\psi_{abc}$. Furthermore, $R_{abcd}$ is expressed in terms of its irreducible components.

Finally, using the kinematic quantities defined in Eq. (64) we finally rewrite Eqs. (113-114) as:

$$\dot{\mathcal{E}}_{\langle f \rangle} = -\mathcal{E}^a \left[\chi h_{fa} - h_{fb}\left(\epsilon U_a a^b + \chi^b{}_a\right)\right] + \mathrm{curl}\mathcal{B}_f - \epsilon\epsilon_{afcd}\mathcal{B}^c U^a a^d + \epsilon h^d{}_f \mathcal{S}_d, \tag{118}$$

$$\dot{\mathcal{B}}_{\langle f \rangle} = -2\mathcal{B}^a \left[\chi h_{fa} - h_{fb}\left(\epsilon U_a a^b + \chi^b{}_a\right)\right] - \mathrm{curl}\mathcal{E}_f + \epsilon\epsilon_{afcd}\mathcal{E}^c U^a a^d - \epsilon h^d{}_f \mathcal{V}_d, \tag{119}$$

where $\mathcal{V}_b h^b{}_v$ and $\mathcal{S}_d h^d{}_v$ are given by (117)-(118). These equations provide the required evolution equations for the components of the auxiliary field $\psi_{abc}$. The hyperbolicity follows by comparison with the Maxwell evolution equations, Eqs. (40)-(41).

### 4.5 Propagation equations for frame coefficients and connection coefficients

We now consider the evolution equations for the tetrad coefficients, $e_a{}^\mu$, and the Ricci rotation coefficients, $\Gamma_b{}^a{}_c$.

The gauge choice $N = e_0 = U$, $N_a = U_a$, $U^a = e_0{}^a = \delta_0{}^a$ together with Eq. (4) readily give

$$\partial_t e_i{}^\mu = (\Gamma_0{}^j{}_i - \Gamma_i{}^j{}_0) e_j{}^\mu + \Gamma_0{}^0{}_i e_0{}^\mu. \tag{120}$$

In order to obtain evolution equations for the Ricci rotation coefficients, one imposes the gauge condition $\Gamma_0{}^j{}_i = 0$ arising from the Fermi propagation of the spatial frame vectors —see [3]. It can be verified that as a consequence of the metric compatibility condition $\Gamma_{a(bc)} = 0$ one only needs to consider evolution equations for the component $\Gamma_q{}^i{}_j$, $\Gamma_0{}^0{}_i$ and $\Gamma_i{}^0{}_j$.

A propagation equation for $\Gamma_q{}^i{}_j$ is readily found from Eq. (7). One has that

$$\partial_t \Gamma_q{}^i{}_j = R^i_{j0q} - \Gamma_k{}^i{}_j \Gamma_q{}^k{}_0 - \Gamma_0{}^i{}_0 \Gamma_q{}^0{}_j + \Gamma_0{}^0{}_j \Gamma_q{}^i{}_0. \tag{121}$$

The propagation equations for $\Gamma_0{}^0{}_i$ and $\Gamma_q{}^0{}_j$ require a more elaborated analysis involving the fluid equations. From Eq. (49) together with the thermodynamic relations (56) we obtain

$$\dot{p} = -\nu_s^2 \nabla^a U_a (\rho + p) - U^b F_b{}^c \nabla^a F_{ac}. \tag{122}$$

Furthermore, using Eq. (50) and Eq. (122) we have

$$\epsilon \nabla_d p = (\rho + p)(\dot{U}_d - U_d \nu_s^2 \nabla^a U_a) - \epsilon \nabla^a F_{ac} F_d{}^c, \tag{123}$$

so that the condition $\nabla_{[a}\nabla_{b]} p = 0$ with Eq. (123) implies

$$0 = \nabla_e \Big((\rho + p)(\dot{U}_d - U_d \nu_s^2 \nabla^a U_a)\Big) - \nabla_d \Big((\rho + p)(\dot{U}_e - U_e \nu_s^2 \nabla^a U_a)\Big) - 2\epsilon \left(F_{[d}{}^c \nabla_{e]} + \nabla_{[e} F_{d]}{}^c\right) \nabla^a F_{ac}. \tag{124}$$

This equation can be conveniently split as the sum of the three terms:

$$\mathcal{J}_{ed} \equiv \mathcal{A}_{ed} + \mathcal{M}_{ed} + \mathcal{F}_{ed} = 0, \tag{125}$$

where

$$\mathcal{A}_{ed} \equiv 2(\rho+p)\nabla_{[e}(\dot{U}_{d]} - U_{d]}\nu_s^2\nabla^a U_a), \tag{126}$$
$$\mathcal{M}_{ed} \equiv 2(\dot{U}_{[d} - U_{[d}\nu_s^2\nabla^a U_{|a|})\nabla_{e]}(\rho+p), \tag{127}$$
$$\mathcal{F}_{ed} \equiv -2\epsilon\left(F_{[d}{}^{c}\nabla_{e]} + \nabla_{[e}F_{d]}{}^{c}\right)\nabla^a F_{ac}. \tag{128}$$

Now, if one regards the velocity of sound $\nu_s$ as function of $n$ and $s$, then from the $(0,i)$ component of Eq. (125) and Eq. (123) we obtain:

$$\begin{aligned} &e_0\Gamma_0{}^0{}_i - \nu_s^2 e_i\Gamma_j{}^j{}_0 + \Gamma_0{}^0{}_j\left(\Gamma_i{}^j{}_0 - \Gamma_0{}^j{}_i\right) - \nu_s^2\Gamma_j{}^j{}_0\Gamma_0{}^0{}_i = \\ &-\frac{\rho+p}{\nu_s^2}\left(\frac{\partial^2 p}{\partial\rho^2}\right)_s\Gamma_j{}^j{}_0\left(\Gamma_0{}^0{}_i + \frac{1}{p+\rho}(\nabla^d F_{de})F_i{}^e\right) - \frac{\beta}{\rho+p}\Gamma_0{}^0{}_i\nabla_0 s + \frac{\alpha}{\rho+p}\nabla_i s\Gamma_j{}^j{}_0 \\ &-\frac{1}{\rho+p}(\Gamma_0{}^0{}_i\gamma_0 - \Gamma_j{}^j{}_0\gamma_i) - \frac{2}{\rho+p}\nabla_{[0}(F_{i]}{}^c\nabla^d F_{dc}), \end{aligned} \tag{129}$$

where

$$\gamma_a \equiv \frac{\nu_s^2+1}{\nu_s^2}F_a{}^c\nabla^d F_{dc}, \tag{130}$$
$$\beta \equiv nT - \frac{1}{\nu_s^2}\frac{\partial p}{\partial s}, \qquad \alpha \equiv (\rho+p)\frac{\partial\nu_s^2}{\partial s} - \left(1 + \frac{n}{\nu_s^2}\frac{\partial\nu_s^2}{\partial n}\right)\frac{\partial p}{\partial s} + \nu_s^2 nT \tag{131}$$

Similarly, from the $(j,k)$ component of Eq. (125) we find that

$$\begin{aligned} &e_k\Gamma_0{}^0{}_j - e_j\Gamma_0{}^0{}_k - \Gamma_0{}^0{}_i\left(\Gamma_k{}^i{}_j - \Gamma_j{}^i{}_k\right) - \nu_s^2\Gamma_i{}^i{}_0\left(\Gamma_k{}^0{}_j - \Gamma_j{}^0{}_k\right) \\ &+\frac{2\beta}{\rho+p}\nabla_{[k}s\,\Gamma_0{}^0{}_{j]} + \frac{2}{\rho+p}\Gamma_0{}^0{}_{[k}\gamma_{j]} + \frac{2}{\rho+p}\nabla_{[k}(F_{j]}{}^c\nabla^d F_{dc}) = 0. \end{aligned} \tag{132}$$

Finally, making use of Eq. (121) in Eq. (129) one obtains

$$\begin{aligned} e_0\Gamma_0{}^0{}_i - \nu_s^2 e_j\Gamma_i{}^j{}_0 &= \Gamma_0{}^0{}_j\left(\Gamma_0{}^j{}_i - \Gamma_i{}^j{}_0\right) + \nu_s^2\Gamma_j{}^j{}_0\Gamma_0{}^0{}_i - \frac{\rho+p}{\nu_s^2}\left(\frac{\partial^2 p}{\partial\rho^2}\right)_s\Gamma_j{}^j{}_0 \\ &+\left(\Gamma_0{}^0{}_i + \frac{1}{p+\rho}(\nabla^d F_{de})F_i{}^e\right) - \frac{\beta}{\rho+p}\Gamma_0{}^0{}_i\nabla_0 s + \frac{\alpha}{\rho+p}\nabla_i s\Gamma_j{}^j{}_0 - \frac{1}{\rho+p}(\Gamma_0{}^0{}_i\gamma_0 - \Gamma_j{}^j{}_0\gamma_i) \\ &-\frac{2}{\rho+p}\nabla_{[0}(F_{i]}{}^c\nabla^d F_{dc}) + \nu_s^2\left(R^j{}_{0ij} + \Gamma_0{}^j{}_0(\Gamma_i{}^0{}_j - \Gamma_j{}^0{}_i) - \Gamma_p{}^j{}_0\Gamma_j{}^p{}_i + \Gamma_j{}^j{}_p\Gamma_i{}^p{}_0\right), \end{aligned} \tag{133}$$

while using

$$\partial_t\Gamma_k{}^0{}_j = e_k\Gamma_0{}^0{}_j + R^0_{j0k} + \Gamma_0{}^0{}_j\Gamma_0{}^0{}_k - \Gamma_i{}^0{}_j\Gamma_k{}^i{}_0 - \Gamma_0{}^0{}_i\Gamma_k{}^i{}_j \tag{134}$$

from Eq. (7) in Eq. (132) gives

$$e_0\Gamma_k{}^0{}_j - e_j\Gamma_0{}^0{}_k = -\Gamma_0{}^0{}_p(\Gamma_j{}^p{}_k - \Gamma_k{}^p{}_j) - \nu_s^2\Gamma_i{}^i{}_0(\Gamma_j{}^0{}_k - \Gamma_k{}^0{}_j) - \frac{2\beta}{\rho+p}\nabla_{[k}s\Gamma_0{}^0{}_{j]}$$
$$-\frac{2}{\rho+p}\Gamma_0{}^0{}_{[k}\gamma_{j]} - \frac{2}{\rho+p}\nabla_{[k}(F_{j]}{}^c\nabla^d F_{dc}) + R^0{}_{j0k} + \Gamma_0{}^0{}_j\Gamma_0{}^0{}_k - \Gamma_i{}^0{}_j\Gamma_k{}^i{}_0 - \Gamma_0{}^0{}_i\Gamma_k{}^i{}_j. \tag{135}$$

In all the previous it is understood that terms of the form $\nabla^d F_{dc}$ are to be replaced using the relation

$$\nabla^d F_{dc} = \psi_c = \rho_{\mathrm{C}} U_c + \sigma E_c, \tag{136}$$

where it is assumed that the charge density, $\rho_{\mathrm{C}}$, is constant multiple of the fluid density, $\rho = \varrho\rho_{\mathrm{C}}$.

**Remark.** Important for the hyperbolicity of the evolution equations is that the term

$$\frac{2}{\rho+p}\nabla_{[a}(F_{b]}{}^c\nabla^d F_{dc}),$$

appearing in Eqs. (129) and Eq. (132), can be rewritten in terms of $(\Gamma_c{}^a{}_d, s_a, \mathcal{E}_a)$ if one uses Eqs. (100) and (136).

## 4.6 Summary of the analysis

In this subsection we summarise our analysis of the evolution equations. Evolution equations for the *independent* components of the vector variable

$$\mathbf{v} = \Big(e_i{}^\mu, \Gamma_0{}^0{}_i, \Gamma_i{}^a{}_b, \hat{E}_{ab}, \hat{B}_{ab}, E_a, B_a, \mathcal{E}_a, \mathcal{B}_a, n, s, s_a\Big), \tag{137}$$

have been constructed. The components of $\Gamma_i{}^j{}_k$ not included in this list are determined by means of gauge conditions and symmetries. By construction, the electric and magnetic parts of the Weyl tensor are tracefree. This symmetry is disregarded in these considerations and is recovered by imposing it on the initial data. It can be shown that if these tensors are initially tracefree, then they will be also tracefree for all later times —see e.g. [2].

The evolution equations for the independent components of the unknowns in (137) and the underlying assumptions are given as follows:

(i) The propagation equation for the tetrad coefficients, $e_i{}^\mu$, is given by Eq. (120) by virtue of the Lagrange condition $U^a = e_0^a = \delta_0^a$. Eq. (120) has the same principal part than the corresponding equation in the uncharged case as described in [3]. It gives rise to a symmetric hyperbolic subsystem of equations.

(ii) The connection coefficients, $\Gamma_0{}^0{}_i$ and $\Gamma_i{}^a{}_b$ are given, respectively, by Eqs. (129) and Eq. (132). In addition, the gauge condition, $\Gamma_0{}^j{}_i = 0$, is assumed. Eq. (121) takes care of $\Gamma_j{}^q{}_k$. As in the case of the equations frame coefficients, Eqs. (129), (132) and (121) have the same principal part as in the uncharged case analysed in [3]. These equations again form a symmetric hyperbolic subsystem of equations.

(iii) The evolution equations for the electric part, $\hat{E}_{ab}$, and the magnetic part, $\hat{B}_{ab}$, of the Weyl tensor are given, respectively, by Eqs. (91) and (88). As mentioned before, the tracefreeness of these tensors is not used to reduce the number of independent components. Thus one has 12 equations for equal number of components. Equations with a principal part of the form of Eqs. (91) and (88) are symmetric hyperbolic independently of the gauge choice —see e.g. [2].

(iv) The propagation equation for the electric part, $E_a$, and the magnetic part, $B_b$, of the Faraday tensor are given, respectively, by Eqs. (40) and (41). The isotropic Ohm law, Eq. (45) has been assume in the deduction of these equations. As in the case of the equations for the electric and magnetic parts of the Weyl tensor, the principal part of these equations is known to be hyperbolic independently of the gauge —again, see e.g. [2].

(v) The evolution equations for the electric part, $\mathcal{E}_a$, and magnetic part, $\mathcal{B}_b$, of the field $\psi_{abc}$, corresponding to the covariant derivative of the Faraday tensor are given, respectively, by Eqs. (118) and (119). These equations involve 24 equations for as many unknowns. Their structure is analogous to that of Eqs. (40) and (41), except that they contain an extra free index. As a consequence, their principal part gives rise to a symmetric hyperbolic subsystem.

(vi) The evolution equation for the particle number density $n$ is given by Eq. (58). This equation is consistent with symmetric hyperbolicity as in the Lagrangian gauge all the derivatives of the flow vector $U^a$ are replaced by connection coefficients, and thus, it contains no derivatives in the spatial directions.

(vii) The evolution equation for the entropy per particle, $s$, is given by Eq. (59) with the understanding that the covariant derivative of the Faraday tensor arising in the right-hand side is to be expressed in terms, either, of the auxiliary field $\psi_{abc}$ or written in terms of the current vector $J_a$ using the inhomogeneous Maxwell equation (30).

(viii) Finally the equation for $s_a \equiv \nabla_a s$ is obtained by covariant differentiating Eq. (130), commuting covariant derivatives and taking into account having in mind Eqs. (136) and, again, (130).

Summarising, Eqs. (40), (41), (58), (88), (91), (118), (119), (120), (121), (129), (132) and (130) provide the desired symmetric hyperbolic evolution system for the Einstein-charged perfect fluid system. This system can be written in the form

$$\mathbf{A}^0 \partial_0 \mathbf{v} - \mathbf{A}^j \partial_j \mathbf{v} = \mathbf{B}\mathbf{v}, \tag{138}$$

where $\mathbf{A}^\mu = \mathbf{A}^0(x^\mu, \mathbf{v})$ are matrix-valued function of the coordinates and the unknowns $\mathbf{v}$, and $\mathbf{A}^0$ is positive definite. The structure of the system (138) ensures the existence of local solutions to the evolution equations. The analysis of whether these solutions give rise to a solution of the full Einstein-charged perfect fluid system requires the analysis of the propagation of the constraint equations. This is a computationally intensive argument which will be omitted here.

### 4.7 Infinitely conductive plasma

A particularly important subcase of our previous analysis is that of an ideal conductive plasma, defined by the condition $E_a = 0$ —ideal magnetohydrodynamics. The hyperbolic problem for this case can be naturally treated as a subcase of the general case in which the Faraday tensor has all the six independent components. In this case the vector variable $\mathbf{v}$ reduces to

$$\mathbf{v} = \{e_i^\mu, \Gamma_0{}^0{}_i, \Gamma_i{}^a{}_b, \hat{E}_{ab}, \hat{B}_{ab}, B_a, \mathcal{E}_a, \mathcal{B}_a, n, s, s_a\}. \tag{139}$$

Notice that although $E_a = 0$, one nevertheless has a non-vanishing electric part of the auxiliary tensor $\psi_{abc}$ as can be seen from $\mathcal{E}_i = -\Gamma_a{}^j{}_d F_{ij} U^d$.

In the infinitely conductive plasma case the system consisting of Eq. (30), Eq. (50), Eq. (51), and Eq. (59) reduces to

$$U_a \nabla^a \rho + (p+\rho)\nabla^a U_a = 0, \tag{140}$$

$$(p+\rho)U^a \nabla_a U^c - \epsilon h^{bc}\nabla_b p - \epsilon(\nabla^a F_{ad})F_b{}^d h^{bc} = 0, \tag{141}$$

$$U^a \nabla_a s = 0, \tag{142}$$

**Remark 1.** As a consequence of Eq. (142) the entropy per particle is constant along the flow lines of the perfect fluid. This fact is translated into Eq. (142), propagation equation for $s$, that implies the propagation equation $\mathcal{L}_U s_a = 0$.

**Remark 2.** Consistent with Ohm's law the source term of the Maxwell equation is in this case simply given by $J_a = \rho_{\mathrm{C}} U_a$.

## 5 Conclusions

In the present article we have revisited the issue of wellposedness initial value problem for the evolution equations of the Einstein-Maxwell-Euler system (a selfgravitating charged perfect fluid). Our analysis assumes a system consisting of a single relativistic perfect fluid obeying to a certain equation of state. The approach followed makes use of the well known $1+3$ tetrad formalism by means of which the various tensorial quantities and equations are projected along the direction of the comoving observer and onto the orthogonal subspace. Following [3,2], we require the timelike vector of the orthonormal frame to follow the matter flow lines (Lagrangian gauge). Moreover, we assume the vector

fields tetrad to be Fermi transported in the direction of $U$, these conditions fix certain components of the connection.

A key feature of our analysis was the introduction of a rank 3 tensor $\psi_{abc}$ corresponding to the covariant derivative of the Faraday tensor $F_{ab}$. The purpose of introducing this tensor was to ensure the symmetric hyperbolicity of the propagation equations for the components of the Weyl tensor. The symmetries of the tensor $\psi_{abc}$ readily allow a decomposition analogous to that of the Faraday tensor into electric and magnetic parts.

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## A On the equations of state

### A.1 Homentropic flows

A fluid is said to be *homeontropic* is $s$ is constant in space and time. In general, the equation of state is given by $\rho = f(n,s)$. Now, if one has an homeontropic flow, the latter can be written as $\rho = f(n)$ —there is no dependence on $s$ as it is constant. Now, if $f$ is a differentiable function of $n$ and $f'(n) \neq 0$, then one can write $n = f^{-1}(\rho)$. It then follows that

$$\begin{aligned} p &= n\frac{\partial \rho}{\partial n} - \rho, \\ &= nf'(n) - \rho, \\ &= nf'(f^{-1}(\rho)) - \rho. \end{aligned}$$

Thus, $p$ is of the form $p = h(\rho)$ —that is, one obtains a barotropic equation of state.

### A.2 Conditions for a barotropic equation of state

In the previous section we have shown that the assumption of an homeontropic flow implies a barotropic equation of state. We now investigate more general situations for which one can have this type of equation of state.

Assume that $p = h(p)$ . Then, from

$$p(n,s) = n\left(\frac{\partial \rho}{\partial n}\right) - \rho(n,s)$$

it follows that

$$h(f(n,s)) = n\left(\frac{\partial f}{\partial n}\right) - f(n,s).$$

The latter can be read as a (possibly non-linear) differential equation for $f(n,s)$. It can be integrated to obtain

$$\exp\left(\int \frac{\mathrm{d}f}{h(f)+f}\right) = g(s)n,$$

where $g(s)$ is an arbitrary function of $s$. This expression can be used to eliminate $n$ from the discussion. In the case of a dust equation of state ($p=0$), the last relation reduces to

$$\rho/n = g(s).$$

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